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Reduction Formula for Fermion Loops and Density Correlations of the 1D Fermi Gas

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Fermion N-loops with an arbitrary number of density vertices $N > d+1$ in d spatial dimensions can be expressed as a linear combination of $(d+1)$-loops with coefficients that are rational functions of external momentum and energy variables. A theorem on symmetrized products then implies that divergences of single loops for low energy and small momenta cancel each other when loops with permuted external variables are summed. We apply these results to the one-dimensional Fermi gas, where an explicit formula for arbitrary N-loops can be derived. The symmetrized N-loop, which describes the dynamical N-point density correlations of the 1D Fermi gas, does not diverge for low energies and small momenta. We derive the precise scaling behavior of the symmetrized N-loop in various important infrared limits.

KEY WORDS: Fermi systems; Feynman amplitudes; density correlations; surface fluctuations.

1. INTRODUCTION

The properties of fermion loops with density vertices (see Fig. 1) play a role in the theory of Fermi systems and various other problems in statistical mechanics. Symmetrized loops, obtained by summing all permutations of the N external energy-momentum variables of a single N-loop, describe dynamical N-point density correlations of a (non-interacting) Fermi gas. Single loops have no direct physical meaning (for $N > 2$), but contribute as subdiagrams of Feynman diagrams in the perturbation expansion of interacting Fermi systems. Symmetrized loops appear as integral kernels in effective actions for interacting Fermi systems, where fermionic degrees of freedom have been eliminated in favor of collective density fluctuations.[1]

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The behavior of symmetrized loops for small energy and momentum variables is particularly important for Fermi systems with long-range interactions, whose Fourier transform is singular for small energy and momentum transfers.[2, 3]

Besides their relevance for interacting electron systems and other fermionic systems in nature, the theory of Fermi systems has also a bearing on various problems in classical statistical mechanics, which can be mapped to an effective Fermi system (gas or interacting). For example, the statistical mechanics of directed lines in two dimensions can be mapped to the quantum mechanics of fermions in one spatial dimension.[4] This mapping has been exploited extensively to study fluctuations of crystal surfaces.[5, 6]

The 2-loop, corresponding to the 2-point density correlation function has been computed long ago in one, two, and three dimensions.[7] Recently, Feldman *et al.*[8] have obtained an exact expression for the N-loop with arbitrary energy and momentum variables in two dimensions. We have evaluated that expression explicitly and analyzed the small energy-momentum limit of the symmetrized loops, showing in particular that infrared divergencies of single loops cancel completely in the sum over permutations.[9]

Most recently, Wagner[10] has published a reduction formula for fermion loops in the static case, where all energy variables are set zero. This formula reduces the N-loop for a d-dimensional Fermi system to a linear combination of $(d+1)$-loops, with coefficients that are rational functions of the momenta. In this work we point out that Wagner's formula and derivation can be easily extended to the case of finite energy variables (Section 3). In the two-dimensional case, the possibility of such an extension is evident from the exact expression for N-loops.[8] The small energy-momentum behavior of symmetrized N-loops can be analyzed by applying a theorem on symmetrized products derived in our work on two-dimensional systems,[9] which we formulate for the general d-dimensional case in Section 4. We apply the reduction formula to a one-dimensional system, where the N-loop can be expressed in terms the 2-loop, which is very easy to compute (Section 5). We finally compute the infrared scaling behavior of symmetrized N-loops in a one-dimensional Fermi system.

2. LOOPS

The amplitude of the N-loop with density vertices, represented by the Feynman diagram in Fig. 1, is given by

$$\Pi_N(q_1,..., q_N) = I_N(p_1,..., p_N) = \int \frac{d^d k}{(2\pi)^d} \int \frac{dk_0}{2\pi} \prod_{j=1}^{N} G_0(k-p_j) \qquad (1)$$

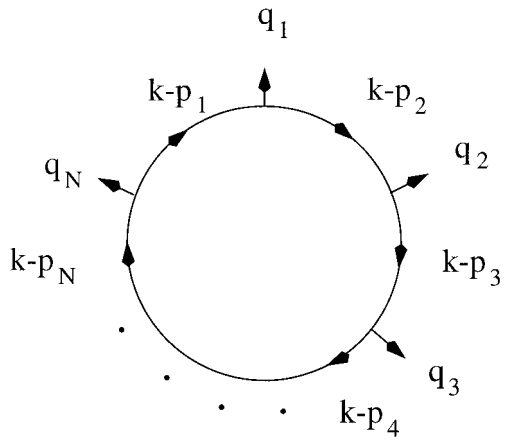

Fig. 1. The N-loop with its energy-momentum labels.

at temperature zero. Here $k=(k_0, \mathbf{k})$, $q_j=(q_{j0}, \mathbf{q}_j)$, and $p_j=(p_{j0}, \mathbf{p}_j)$ are $(d+1)$-dimensional energy-momentum vectors. We use natural units, i.e. $h=1$. The variables q_j and p_j are related by the linear transformation

$$q_j = p_{j+1} - p_j \quad , \quad j=1,..., N \tag{2}$$

where $p_{N+1} \equiv p_1$. Energy and momentum conservation at all vertices yields the restriction $q_1 + \cdots + q_N = 0$. The variables $q_1,..., q_N$ fix $p_1,..., p_N$ only up to a constant shift $p_j \mapsto p_j + p$. Setting $p_1 = 0$, one gets

$$\begin{aligned} p_2 &= q_1 \\ p_3 &= q_1 + q_2 \\ &\vdots \\ p_N &= q_1 + q_2 + \cdots + q_{N-1} \end{aligned} \tag{3}$$

We use the imaginary time representation, with a non-interacting propagator

$$G_0(k) = \frac{1}{ik_0 - (\varepsilon_{\mathbf{k}} - \mu)} \tag{4}$$

where $\varepsilon_{\mathbf{k}}$ is the dispersion relation and μ the chemical potential of the system. For a continuum (not lattice) Fermi system the dispersion relation

is $\varepsilon_{\mathbf{k}} = \mathbf{k}^2/2m$, where m is the fermion mass. The k_0-integral in Eq. (1) can be easily carried out using the residue theorem; one obtains[8]

$$I_N(p_1,..., p_n) = \sum_{i=1}^{N} \int_{|\mathbf{k}-\mathbf{p}_i|<k_F} \frac{d^d k}{(2\pi)^d} \left(\prod_{\substack{j=1 \\ j\neq i}}^{n} f_{ij}(\mathbf{k}) \right)^{-1} \tag{5}$$

where $f_{ij}(\mathbf{k}) = i(p_{i0} - p_{j0}) + \varepsilon_{\mathbf{k}-\mathbf{p}_i} - \varepsilon_{\mathbf{k}-\mathbf{p}_j}$.

The 2-loop $\Pi_2(q, -q) \equiv \Pi(q)$ is known as polarization insertion or particle-hole bubble, and has a direct physical meaning: $\Pi(q)$ is the dynamical density-density correlation function of a non-interacting Fermi system.[11] For $N > 2$, the N-loop is not a physical quantity, but the *symmetrized* N-loop

$$\Pi_N^S(q_1,..., q_N) = \mathscr{S}\Pi_N(q_1,..., q_N) = \frac{1}{N!} \sum_P \Pi_N(q_{P1},..., q_{PN}) \tag{6}$$

where the symmetrization operator $\mathscr{S}$ imposes summation over all permutations of $q_1,..., q_N$, is proportional to the (connected) dynamical N-point density correlation function:

$$\langle \rho(q_1),..., \rho(q_N) \rangle_{con} = (-1)^{N-1} (N-1)!\, \Pi_N^S(q_1,..., q_N) \tag{7}$$

Here $\rho(q)$ is the Fourier transform of the particle density operator. Equation (7) is easily verified by applying Wick's theorem.[11] Note that Wick's theorem yields a sum of $(N-1)!$ distinct loops with non-equivalent permutations of $q_1,..., q_N$, while the sum in Eq. (6) includes cyclic permutations which produce N equivalent copies of each loop.

3. REDUCTION FORMULA

We now state the *reduction formula* that reduces the N-loop for a d-dimensional system with $N > d+1$ to a linear combination of $(d+1)$-loops with coefficients that are explicitly computable rational functions of momentum and energy variables. This formula is a straightforward generalization of a result derived recently by Wagner[10] for the static case $p_{j0} = 0$.

Let $p_1,..., p_N$ be such that for each tupel of integers $\mathbf{j} = (j_1,..., j_{d+1})$ with $1 \leqslant j_1 < \cdots < j_{d+1} \leqslant N$, the complex d-dimensional vectors $\mathbf{d}^{\mathbf{j}}$ determined by the linear equations

$$f_{j_1 j_r}(\mathbf{d}^{\mathbf{j}}) = i(p_{j_1 0} - p_{j_r 0}) + \frac{1}{2m}(\mathbf{p}_{j_1}^2 - \mathbf{p}_{j_r}^2) + \frac{1}{m}(\mathbf{p}_{j_r} - \mathbf{p}_{j_1}) \cdot \mathbf{d}^{\mathbf{j}} = 0 \tag{8}$$

for $r = 2,..., d+1$ are well-defined and unique. Suppose that for $n = 1,..., N$ with $n \neq j_1,..., j_{d+1}$ the numbers

$$f_n^{\mathbf{j}} := f_{j_r n}(\mathbf{d}^{\mathbf{j}}) = i(p_{j_r 0} - p_{n0}) + \frac{1}{2m}(\mathbf{p}_{j_r}^2 - \mathbf{p}_n^2) + \frac{1}{m}(\mathbf{p}_n - \mathbf{p}_{j_r}) \cdot \mathbf{d}^{\mathbf{j}} \tag{9}$$

are non-zero. Then

$$I_N(p_1,..., p_N) = \sum_{\substack{j_1,..., j_{d+1} \\ 1 \leqslant j_1 < \cdots < j_{d+1} \leqslant N}} \left(\prod_{\substack{n=1 \\ n \neq j_1,..., j_{d+1}}}^{N} \frac{1}{f_n^{\mathbf{j}}} \right) I_{d+1}(p_{j_1},..., p_{j_{d+1}}) \tag{10}$$

Note that the numbers $f_{j_r n}(\mathbf{d}^{\mathbf{j}})$ with $r = 1,..., d+1$ are all equal, as a consequence of Eq. (8). The vector $\mathbf{d}^{\mathbf{j}}$ is uniquely defined if the vectors $\mathbf{p}_{j_r} - \mathbf{p}_{j_1}$, where $r = 2,..., d+1$, are linearly independent. The real part of $\mathbf{d}^{\mathbf{j}}$ is the center of the uniquely defined circumscribing sphere through the points $\mathbf{p}_{j_1},..., \mathbf{p}_{j_{d+1}}$ in d-dimensional euclidean space. In contrast to $\mathbf{d}^{\mathbf{j}}$, the numbers $f_n^{\mathbf{j}}$ are invariant under a shift $p_j \mapsto p_j + p$ and can thus be expressed in terms of the variables $q_1,..., q_N$.

The proof of the above reduction formula, a simple generalization of the proof given by Wagner[10] for the static case, is presented in the Appendix.

4. SYMMETRIZED PRODUCTS

Symmetrized loops are obtained by summing over all permutations of external energy-momentum variables $q_1,..., q_N$ as in Eq. (6). Up to a trivial constant, symmetrized N-loops are the connected N-point density correlation functions of the Fermi gas. The behavior of these functions in the infrared limit $q_j \to 0$ determines the long-distance (in space and time) density correlations, and is a crucial ingredient for power-counting of contributions to effective actions for collective density fluctuations. We will consider two important scaling limits:

(i) *small energy-momentum* limit $\lim_{\lambda \to 0} \Pi_N^S(\lambda q_1,..., \lambda q_N)$,

(ii) *dynamical* limit $\lim_{\lambda \to 0} \Pi_N^S[(q_{10}, \lambda \mathbf{q}_1),..., (q_{N0}, \lambda \mathbf{q}_N)]$.

Single N-loops diverge generally (for almost all choices of $q_1,..., q_N$) as λ^{2-N} in the small energy-momentum limit, which is what one would expect from simple power-counting applied to the integral (1). A notable exception is the socalled *static* limit, where the momenta $\mathbf{q}_j$ tend to zero after all energy variables q_{j0} have vanished. In that case one obtains a unique finite limit $\Pi_N \to [(-1)^{N-1}/(N-1)!][d^{N-2}/d\varepsilon^{N-2}]\, D(\varepsilon)|_{\varepsilon=\mu}$, where $D(\varepsilon)$ is the

density of states.[12] In the following we will show that systematic cancellations occur in the sum over permutations in the general small energy-momentum limit and also in the dynamical limit.

The factor multiplying the $(d+1)$-loops in the reduction formula can be written as

$$\mathscr{F}^{\mathbf{j}} := \prod_{\substack{n=1 \\ n \neq j_1,\ldots,j_{d+1}}}^{N} \frac{1}{f_n^{\mathbf{j}}} = \prod_{r=1}^{d+1} F_r^{\mathbf{j}} \tag{11}$$

where

$$F_r^{\mathbf{j}} = \begin{cases} F_r^{\mathbf{j}}(q_{j_r}, q_{j_r+1},\ldots, q_{j_{r+1}-1}) = \prod_{n=j_r+1}^{j_{r+1}-1} \frac{1}{f_n^{\mathbf{j}}} & \text{for} \quad j_{r+1} > j_r + 1 \\ 1 & \text{for} \quad j_{r+1} = j_r + 1 \end{cases} \tag{12}$$

Here $j_{d+2} \equiv j_1$, i.e. for $r = d+1$ the index n runs from $j_{d+1}+1$ to N and then from 1 to $j_1 - 1$. Note that $F_r^{\mathbf{j}}$ depends also on differences of the energy-momentum variables $p_{j_1},\ldots, p_{j_{d+1}}$, besides the explicitly written arguments. As a product of $M_r = j_{r+1} - j_r - 1$ factors $(f_n^{\mathbf{j}})^{-1}$, $F_r^{\mathbf{j}}$ diverges as λ^{-M_r} in the small energy-momentum limit, since each $f_n^{\mathbf{j}}$ vanishes linearly. We define a symmetrized product

$$S_r^{\mathbf{j}}(k_1,\ldots, k_{M_r+1}) = \frac{1}{(M_r+1)!} \sum_P F_r^{\mathbf{j}}(k_{P1},\ldots, k_{P(M_r+1)}) \tag{13}$$

where all permutations of $k_1,\ldots, k_{M_r+1}$ are summed. According to the following theorem, the symmetrized product $S_r^{\mathbf{j}}$ can be expressed such that the cancellations of singularities in the infrared limit become obvious.

Factorization Theorem. The symmetrized product $S_r^{\mathbf{j}}$ can be written as $m^{M_r}/(M_r+1)!$ times a sum over fractions with numerators

$$(\mathbf{k}_{\sigma_1} \cdot \mathbf{k}_{\sigma_1'})(\mathbf{k}_{\sigma_2} \cdot \mathbf{k}_{\sigma_2'})(\mathbf{k}_{\sigma_{M_r}} \cdot \mathbf{k}_{\sigma_{M_r}'}) \tag{14}$$

where $\sigma_i \neq \sigma_i'$ and $M_r = j_{r+1} - j_r - 1$, and products of $2M_r$ functions $f^{\mathbf{j}}$ as denominators. The functions $f^{\mathbf{j}}$ have the form

$$f^{\mathbf{j}}(p, p') = i(p_0 - p_0') + \frac{1}{2m}(\mathbf{p}^2 - \mathbf{p}'^2) + (\mathbf{p}' - \mathbf{p}) \cdot \mathbf{d}^{\mathbf{j}} \tag{15}$$

where $p = p_{j_r}$ and $p' = p_{j_r} + (\textit{partial sum of } k_1,\ldots, k_{M_r+1})$. In each numerator, each momentum variable $k_1,\ldots, k_{M_r+1}$ appears at least once as a factor in one of the scalar products.

For example, in the simplest case $M_r = 1$ one obtains

$$F_r^{\mathbf{j}}(k_1, k_2) + F_r^{\mathbf{j}}(k_2, k_1) = \frac{m(\mathbf{k}_1 \cdot \mathbf{k}_2)}{f^{\mathbf{j}}(p_{j_r}, p_{j_r} + k_1)\, f^{\mathbf{j}}(p_{j_r}, p_{j_r} + k_2)} \tag{16}$$

The factorization theorem has been derived recently[9] in the context of two-dimensional systems. The proof provides a concrete algorithm leading to the factorized expression. Since the algorithm is actually independent of the dimensionality of the system, we will not repeat the derivation here.

The infrared scaling behavior of $S_r^{\mathbf{j}}$ follows directly:

(i) $S_r^{\mathbf{j}}$ is *finite* (of order one) and *real* in the small energy-momentum limit.

(ii) $S_r^{\mathbf{j}}$ vanishes as λ^{2M_r} in the dynamical limit.

To see this, note that the functions $f^{\mathbf{j}}(p, p')$ vanish linearly in the small energy-momentum limit, and are purely imaginary to leading order in λ, while they remain finite in the dynamical limit.

The symmetrized product is thus much smaller for small energy and momentum variables than each single term, namely by a factor λ^{M_r} in the small energy-momentum limit, and even by a factor λ^{2M_r} in the dynamical limit. This result holds in any dimension d.

5. ONE-DIMENSIONAL SYSTEMS

We now apply the general results from Sections 3 and 4 to one-dimensional systems,[13] where particularly simple expressions can be obtained. We consider first single, then symmetrized loops.

A. Single Loops

In one dimension, the reduction formula (10) reduces N-loops to linear combinations of 2-loops:

$$I_N(p_1, \ldots, p_N) = \sum_{\substack{j_1, j_2 \\ 1 \leqslant j_1 < j_2 \leqslant N}} \left[\prod_{\substack{n=1 \\ n \neq j_1, j_2}}^{N} \frac{1}{f_n^{\mathbf{j}}} \right] I_2(p_{j_1}, p_{j_2}) \tag{17}$$

where $d^{\mathbf{j}}$ is given explicitly by

$$d^{\mathbf{j}} = \frac{1}{2}(p_{j_1 1} + p_{j_2 1}) + im \frac{p_{j_1 0} - p_{j_2 0}}{p_{j_1 1} - p_{j_2 1}} \tag{18}$$

and

$$f_n^{\mathbf{i}} = -\frac{1}{2m}(p_{n1} - p_{j_1 1})(p_{n1} - p_{j_2 1}) + i(p_{j_1 0} - p_{n0}) + i(p_{n1} - p_{j_1 1})\frac{p_{j_1 0} - p_{j_2 0}}{p_{j_1 1} - p_{j_2 1}} \tag{19}$$

Here p_{n1} and $p_{j_r 1}$ are the one-dimensional momentum components of the energy-momentum vectors $p_n = (p_{n0}, p_{n1})$ and $p_{j_r} = (p_{j_r 0}, p_{j_r 1})$, respectively. The 2-loop can be computed very easily, the result being

$$I_2(p_{j_1}, p_{j_2}) = \frac{m}{\pi}\frac{1}{p_{j_1 1} - p_{j_2 1}} \log\left|\frac{k_F - \alpha_{j_1 j_2}}{k_F + \alpha_{j_1 j_2}}\right| \tag{20}$$

where

$$\alpha_{j_1 j_2} = \frac{1}{2}(p_{j_1 1} - p_{j_2 1}) + im\frac{p_{j_1 0} - p_{j_2 0}}{p_{j_1 1} - p_{j_2 1}} \tag{21}$$

We have thus obtained an explicit expression in terms of elementary functions for N-loops in one dimension. One may easily perform an analytic continuation to real (instead of imaginary) energy variables, $ip_{j0} \mapsto \varepsilon_j$, in the above expressions to analyze, for example, the non-linear dynamical density response of the Fermi gas.

In the zero energy limit $p_{j0} \to 0$ one obtains the simple result

$$\lim_{\substack{p_{j0}\to 0 \\ j=1,\ldots,N}} I_N(p_1,\ldots,p_N) = \sum_{\substack{j_1, j_2 \\ 1 \leqslant h_1 < j_2 \leqslant N}} \left[\prod_{\substack{n=1 \\ n \neq j_1, j_2}}^{N} \frac{-2m}{(p_{n1} - p_{j_1 1})(p_{n1} - p_{j_2 1})}\right] \times \frac{m}{\pi(p_{j_1 1} - p_{j_2 1})} \log\left|\frac{2k_F - (p_{j_1 1} - p_{j_2 1})}{2k_F + (p_{j_1 1} - p_{j_2 1})}\right| \tag{22}$$

Note that the above expression has a finite limit for $p_{j1} \to 0$, although each contribution to the sum diverges.

B. Symmetrized Loops

It is well known that for a *linearized* dispersion relation $\varepsilon_k = v_F(|k| - k_F)$, as in the one-dimensional Luttinger model, the symmetrized N-loop $\Pi_N^S(q_1,\ldots,q_N)$ vanishes identically for $N > 2$ even for finite q_j with sufficiently small momenta q_{j1}.[14] We now analyze the infrared behavior of symmetrized N-loops in a one-dimensional system with the usual quadratic

dispersion relation. Symmetrizing the reduction formula, we can write symmetrized loops as

$$\Pi_N^S(q_1,..., q_N) = \mathscr{S} \sum_{\substack{j_1, j_2 \\ 1 \leqslant j_1 < j_2 \leqslant N}} S_1^{\mathbf{j}} S_2^{\mathbf{j}} I_2(p_{j1}, p_{j2}) \tag{23}$$

where $\mathscr{S}$ is the symmetrization operator introduced in Section 2 and $S_1^{\mathbf{j}}$ and $S_2^{\mathbf{j}}$ are the symmetrized products defined in Section 4. Note that first symmetrizing partially (with respect to a subset of variables, as in the products $S_r^{\mathbf{j}}$) and then completely (by applying $\mathscr{S}$) yields the same result as symmetrizing everything just once.

We can now easily derive the scaling behavior of Π_N^S in the small energy-momentum and dynamical limit, respectively. The 2-loop $\Pi(q_1) \equiv \Pi_2(q_1, -q_1) = I_2(0, q_1)$ tends to the finite value

$$\Pi(\lambda q_1) \to -\frac{1}{\pi v_F} \frac{1}{1 + [q_{10}/(v_F q_{11})]^2} \tag{24}$$

in the small energy-momentum limit and vanishes quadratically as

$$\Pi(q_{10}, \lambda q_{11}) \to -\frac{v_F}{\pi} \frac{q_{11}^2}{q_{10}^2} \lambda^2 \tag{25}$$

in the dynamical limit, where $v_F = k_F/m$ is the Fermi velocity. The same behavior is found for the 2-loop with a linearized ε_k. Since $S_1^{\mathbf{j}}$ and $S_2^{\mathbf{j}}$ are both finite in the small energy momentum limit, the symmetrized N-loop remains finite, too:

$$\Pi_N^S(\lambda q_1,..., \lambda q_N) = O(1) \qquad \text{for} \quad \lambda \to 0 \tag{26}$$

Only in the static case $q_{j0} = 0$ each single loop Π_N has a finite limit for $q_{j1} \to 0$, while in general the above result is due to systematic cancellations of infrared divergencies. In the dynamical limit the product $S_1^{\mathbf{j}} S_2^{\mathbf{j}}$ vanishes as $\lambda^{2M_1 + 2M_2}$ where $M_1 + M_2 = N - 2$, such that

$$\Pi_N^S[(q_{10}, \lambda q_{11}),..., (q_{N0}, \lambda q_{N1})] = O(\lambda^{2N-2}) \qquad \text{for} \quad \lambda \to 0 \tag{27}$$

The same scaling behavior has been found previously for two-dimensional systems.[9]

6. CONCLUSION

We have derived a formula that reduces the evaluation of fermion loops with N density vertices in d dimensions to the computation of loops

with only $d+1$ vertices. This was obtained by a straightforward extension of a recent result by Wagner[10] for the zero energy limit to arbitrary energy variables. Using a theorem about symmetrized products, we have shown that infrared divergencies of single loops cancel to a large extent when permutations of external energy-momentum variables are summed. The symmetrized N-loop, which is proportional to the N-point density correlation function of the Fermi gas, is thus generally much smaller in the infrared limit than unsymmetrized loops. For one-dimensional systems, we have obtained an explicit expression for arbitrary N-loops in terms of elementary functions of the energy-momentum variables. We have shown that symmetrized loops do not diverge for low energies and small momenta. In the dynamical limit, where momenta scale to zero at fixed energy variables, the symmetrized N-loop vanishes as the $(2N-2)$th power of the scale parameter.

We finally outline some applications of our results.

Evaluation of Feynman diagrams. Analytical results for loops are of course useful for computing Feynman diagrams containing fermion loops as subdiagrams. The number of energy-momentum variables that remain to be integrated (analytically or numerically) is thus reduced. In particular, the mutual cancellation of contributions associated with different permutations of energy-momentum transfers entering a loop can be treated analytically, avoiding numerical "minus-sign" problems.

Effective actions. Effective actions for interacting Fermi systems, where the fermionic degrees of freedom have been eliminated in favor of collective density fluctuations, contain symmetrized N-loops as kernels.[1] A good control of the infrared behavior of these kernels is essential for assessing the relevance of non-Gaussian terms in the effective action, especially in the presence of long-range interactions. In one-dimensional systems one can use our results to compute the scaling dimensions of corrections to the leading low-energy behavior of Luttinger liquids[15] by analyzing the non-quadratic corrections in the bosonized action.

Surface fluctuations. Some models of surface fluctuations lead to the statistical mechanics of directed lines in two dimensions, which can be mapped to the quantum mechanics of fermions in one spatial dimension.[5, 6] Most recently, Prähofer and Spohn[16] have shown that the probability distribution of height fluctuations in such models is Gaussian at long distances on the surface. For this result it was enough to establish that symmetrized N-loops in the associated Fermi system are less singular than the naive power-counting estimate. Our result Eq. (26) yields the precise scaling dimension of non-Gaussian terms, and implies in particular that high order corrections vanish very rapidly at long distances.

APPENDIX A. PROOF OF REDUCTION FORMULA

Following Wagner's[10] derivation for the static case, we prove the reduction formula (10) by applying the following many-dimensional version of Lagrange's interpolation formula:

Lemma 1. Suppose that $1 \leqslant d+1 < N$ and the $(d+1)$-dimensional complex vectors $\mathbf{a}_1,..., \mathbf{a}_N$ are such that $\mathbf{a}_{j_1},..., \mathbf{a}_{j_{d+1}}$ as well as $(\mathbf{a}_{j_1} - \mathbf{a}_n),..., (\mathbf{a}_{j_{d+1}} - \mathbf{a}_n)$ are linearly independent for pairwise different indices $j_1,..., j_{d+1}$, $n \in \{1,..., N\}$. For $\mathbf{j} = (j_1,..., j_{d+1})$ with $1 \leqslant j_1 < \cdots < j_{d+1} \leqslant N$ determine the complex $(d+1)$-dimensional vector $\mathbf{z}^{\mathbf{j}}$ by the system of linear equations $\mathbf{a}_{j_r} \cdot \mathbf{z}^{\mathbf{j}} = 1$ for $r = 1,..., d+1$. Then each complex homogeneous polynomial $P(z_0, z)$ of degree $N-(d+1)$ in the $d+2$ variables z_0, $\mathbf{z} = (z_1,..., z_{d+1})$ can be written as

$$P(z_0, \mathbf{z}) = \sum_{\substack{j_1,..., j_{d+1} \\ 1 \leqslant j_1 < \cdots < j_{d+1} \leqslant N}} P(1, \mathbf{z}^{\mathbf{j}}) \prod_{\substack{n=1 \\ n \neq j_1,..., j_{d+1}}}^{N} \frac{(z_0 - \mathbf{a}_n \cdot \mathbf{z}) \det(\mathbf{a}_{j_1},..., \mathbf{a}_{j_{d+1}})}{\det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \mathbf{a}_n & \mathbf{a}_{j_1} & \cdots & \mathbf{a}_{j_{d+1}} \end{pmatrix}} \tag{28}$$

where the vectors $\mathbf{a}_1,..., \mathbf{a}_N$ enter the determinants as column vectors. For a proof, see ref. 17.

We apply the above lemma to the polynomial $P(z_0, \mathbf{z}) = z_1^{N-(d+1)}$ and

$$\mathbf{a}_n = \begin{pmatrix} -i(k_0 - p_{n0}) + \xi_{\mathbf{p}_n} \\ (\mathbf{k} - 2\mathbf{p}_n)/\sqrt{2m} \end{pmatrix} \tag{29}$$

where $\xi_{\mathbf{p}} = \mathbf{p}^2/(2m) - \mu$. Since $P(1, \mathbf{z}^{\mathbf{j}}) = (z_1^{\mathbf{j}})^{N-(d+1)}$ and $\mathbf{a}_{j_r} \cdot \mathbf{z}^{\mathbf{j}} = 1$ for $r = 1,..., d+1$, Cramer's rule yields

$$\begin{aligned} P(1, \mathbf{z}^{\mathbf{j}}) & \prod_{\substack{n=1 \\ n \neq j_1,..., j_{d+1}}}^{N} \det(\mathbf{a}_{j_1},..., \mathbf{a}_{j_{d+1}}) \\ &= \prod_{\substack{n=1 \\ n \neq j_1,..., j_{d+1}}}^{N} z_1^{\mathbf{j}} \det(\mathbf{a}_{j_1},..., \mathbf{a}_{j_{d+1}}) \\ &= \prod_{\substack{n=1 \\ n \neq j_1,..., j_{d+1}}}^{N} \det \begin{pmatrix} 1 & \cdots & 1 \\ \dfrac{\mathbf{k} - 2\mathbf{p}_{j_1}}{\sqrt{2m}} & \cdots & \dfrac{\mathbf{k} - 2\mathbf{p}_{j_{d+1}}}{\sqrt{2m}} \end{pmatrix} \\ &= \prod_{\substack{n=1 \\ n \neq j_1,..., j_{d+1}}}^{N} (-\sqrt{2/m})^d \det(\mathbf{p}_{j_2} - \mathbf{p}_{j_1},..., \mathbf{p}_{j_{d+1}} - \mathbf{p}_{j_1}) \end{aligned} \tag{30}$$

In the last step we have subtracted the first column of the determinant from all the others and then applied Laplace's theorem. We now evaluate the denominator in (28),

$$D = \det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ -i(k_0 - p_{n0}) + \xi_{\mathbf{p}_n} & -i(k_0 - p_{j_1 0}) + \xi_{\mathbf{p}_{j_1}} & \cdots & -i(k_0 - p_{j_{d+1} 0}) + \xi_{\mathbf{p}_{j_{d+1}}} \\ (\mathbf{k} - \mathbf{p}_n)/\sqrt{2m} & (\mathbf{k} - \mathbf{p}_{j_1})/\sqrt{2m} & \cdots & (\mathbf{k} - \mathbf{p}_{j_{d+1}})/\sqrt{2m} \end{pmatrix} \tag{31}$$

Subtracting the first column from all the others and applying Laplace's theorem yields

$$D = \det \begin{pmatrix} f_{j_1 n}(0) & \cdots & f_{j_{d+1} n}(0) \\ \sqrt{2/m}\,(\mathbf{p}_n - \mathbf{p}_{j_1}) & \cdots & \sqrt{2/m}\,(\mathbf{p}_n - \mathbf{p}_{j_{d+1}}) \end{pmatrix} \tag{32}$$

Adding $\mathbf{d}^{\mathbf{j}} \cdot (\mathbf{p}_n - \mathbf{p}_{j_r})/m$ to the rth matrix element in the first row (adding thus multiples of the other rows to the first one) one obtains

$$\begin{aligned} D &= \det \begin{pmatrix} f_{j_1 n}(\mathbf{d}^{\mathbf{j}}) & \cdots & f_{j_{d+1} n}(\mathbf{d}^{\mathbf{j}}) \\ \sqrt{2/m}\,(\mathbf{p}_n - \mathbf{p}_{j_1}) & \cdots & \sqrt{2/m}\,(\mathbf{p}_n - \mathbf{p}_{j_{d+1}}) \end{pmatrix} \\ &= (\sqrt{2/m})^d\, f_n^{\mathbf{j}} \det \begin{pmatrix} 1 & \cdots & 1 \\ \mathbf{p}_n - \mathbf{p}_{j_1} & \cdots & \mathbf{p}_n - \mathbf{p}_{j_{d+1}} \end{pmatrix} \end{aligned} \tag{33}$$

Subtracting the first column from all others and applying Laplace's theorem once again one obtains

$$D = (-\sqrt{2/m})^d\, f_n^{\mathbf{j}} \det(\mathbf{p}_{j_2} - \mathbf{p}_{j_1}, \ldots, \mathbf{p}_{j_{d+1}} - \mathbf{p}_{j_1}) \tag{34}$$

Equation (30) and Eq. (34) yield

$$P(1, \mathbf{z}^{\mathbf{j}}) \prod_{\substack{n=1 \\ n \neq j_1, \ldots, j_{d+1}}}^{N} \frac{\det(\mathbf{a}_{j_1}, \ldots, \mathbf{a}_{j_{d+1}})}{\det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \mathbf{a}_n & \mathbf{a}_{j_1} & \cdots & \mathbf{a}_{j_{d+1}} \end{pmatrix}} = \prod_{\substack{n=1 \\ n \neq j_1, \ldots, j_{d+1}}}^{N} \frac{1}{f_n^{\mathbf{j}}} \tag{35}$$

We now set $z_0 = 0$ and $\mathbf{z} = (1, \mathbf{k}/\sqrt{2m})$, such that

$$z_0 - \mathbf{a}_n \cdot \mathbf{z} = i(k_0 - p_{n0}) - \xi_{\mathbf{k} - \mathbf{p}_n} = G_0^{-1}(k - p_n) \tag{36}$$

With this choice of variables the above lemma thus yields the algebraic identity

$$1 = \sum_{\substack{j_1,\ldots,j_{d+1} \\ 1 \leqslant j_1 < \cdots < j_{d+1} \leqslant N}} \prod_{\substack{n=1 \\ n \neq j_1,\ldots,j_{d+1}}}^{N} \frac{1}{f^{\mathbf{j}}_n} G_0^{-1}(k-p_n) \tag{37}$$

Multiplying this equation with $\prod_{j=1}^{N} G_0(k-p_j)$ and integrating over k one finally obtains the reduction formula Eq. (10).

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